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Generalised hypergeometric series ${}^N F\ (x_1, \ldots, x_N)$ arising in physical and quantum chemical applications

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Abstract. Functions ${}^N F\ (x_1, \ldots, x_N)$ which are a straightforward generalisation of standard hypergeometric functions of N variables are introduced. A convenient operator representation is established for these functions which permits us to consider, in a uniform and simple way, standard hypergeometric functions (generalised hypergeometric series of one variable, Appell, Kampé de Fériet and Lauricella functions, etc) as well as many other nameless hypergeometric series arising in physical and quantum chemical applications. Generating functions, generalisations of Bailey and Chaundy expansions, recurrence relations, reduction rules and some other topics are discussed.

1. Introduction

The hypergeometric functions frequently arise in integration of differential equations. There are, however, other important cases which lead to these functions as well. For example, many auxiliary algebraic and integral transformations in various physical models give rise to these functions. In these latter cases only a restricted problem of calculating or establishing some special properties of such functions has usually arisen. The well known examples are: the generalised hypergeometric series of one variable, ${}_pF_q$, for example, the confluent hypergeometric Kummer function, ${}_1F_1$; the Gauss function, ${}_2F_1$; the Clausen function, ${}_3F_2$, in terms of which the Clebsch–Gordan coefficients are expressed; the ${}_4F_3$ function to which the Racah coefficients are related (Varshalovich *et al* 1975), etc. In a number of physical applications hypergeometric functions of many variables arise, for example, the Appell functions in quantum mechanics of atomic systems (Brown 1967), the Lauricella functions in the hyperspherical harmonics model (Erdélyi 1953, § 12.5), and so on. Though a vast literature is devoted to the hypergeometric series, nevertheless, versatile types of functions and a rather wide range of questions arising in applications lead to some difficulties in analysing necessary properties even in the case of standard functions.

A vast field of application of the hypergeometric functions is represented also by the quantum chemistry problems, in particular, the problem of multicentre matrix elements whose calculation is the main difficulty in applications of variational methods to molecular systems. Here, in a number of cases the functions arise which, though similar to the standard ones, do not coincide with the latter to the last term. Certain difficulties are encountered when analysing these functions, since they have few recorded properties (recurrence relations, expansions, addition theorems etc). The derivation of such properties by means of the 'standard' methods, which are far from elementary ones, is often rather cumbersome and, at the same time, often an ungrateful job, since relations obtained for a specific function may be directly transferred to other functions of similar structure only in a few cases. Hence, one meets with the necessity of repeated, tiresome calculations *ad hoc*.

In this connection, it seems expedient to extend a class of the hypergeometric functions to such a minimum range that would embrace as many standard functions as possible, as well as other functions of similar structure, arising in applications, which could be studied in a uniform and simple way.

2. Function $^N F$: definition, relations to other functions, terminology

For a standard generalised hypergeometric series of one variable (GHS-1) we shall use notation which differs slightly from the conventional designation ${}_pF_q$ (Bailey 1935a, Erdélyi 1953, Luke 1975):

$$F_q^p\left[{}^{\boldsymbol{a}:x}_{\boldsymbol{c}}\right]=\sum_{i=0}^{\infty}\frac{(\boldsymbol{a})_i}{(\boldsymbol{c})_i}\frac{x^i}{i!}. \tag{1}$$

Symbols $\boldsymbol{a}$, $\boldsymbol{c}$ denote the sets of numbers

$$\boldsymbol{a}=a^{1\ldots p}=(a^1,\ldots,a^p), \qquad \boldsymbol{c}=c^{1\ldots q}=(c^1,\ldots,c^q), \tag{2}$$

$$(\boldsymbol{a})_i=(a^{1\ldots p})_i=\prod_{r=1}^{p}(a^r)_i, \qquad (\boldsymbol{c})_i=(c^{1\ldots q})_i=\prod_{t=1}^{q}(c^t)_i, \tag{3}$$

where $(a)_i=\Gamma(a+i)/\Gamma(a)$ is a Pochhammer symbol. Introduce the function $^N F$ $(x_1,\ldots,x_N)$,

$$^N F^{p_0,p_1\ldots p_N}_{q_0,q_1\ldots q_N}\left[{}^{\boldsymbol{a}_0:\boldsymbol{a}_1,\ldots,\boldsymbol{a}_N;x_1,\ldots,x_N}_{\boldsymbol{c}_0:\boldsymbol{c}_1,\ldots,\boldsymbol{c}_N}\right]=\sum_{i_1,\ldots,i_N}\frac{(\boldsymbol{a}_0)_{i_1+\ldots+i_N}}{(\boldsymbol{c}_0)_{i_1+\ldots+i_N}}\prod_{s=1}^{N}\frac{(\boldsymbol{a}_s)_{i_s}}{(\boldsymbol{c}_s)_{i_s}}\frac{x_s^{i_s}}{i_s!}, \tag{4}$$

which will be referred to as a generalised hypergeometric series of N variables (GHS-N). 'Vectors' $\boldsymbol{a}_\sigma$, $\boldsymbol{c}_\sigma$ $(\sigma=0,1,\ldots,N)$ have dimensions p_σ, q_σ (see equations (2), (3)), respectively. In contrast to 'individual' (i) parameters $\boldsymbol{a}_s$, $\boldsymbol{c}_s$ $(s=1,2,\ldots,N)$, we shall call $\boldsymbol{a}_0$, $\boldsymbol{c}_0$ parameters 'gluons', or gluing (g) factors, since in their absence (the case $p_0=q_0=0$ which will be symbolised by a standard sign $*$) the series (4) breaks up into the product of N GHS-1:

$$^N F^{0,p_1\ldots p_N}_{0,q_1\ldots q_N}\left[{}^{*;\boldsymbol{a}_1,\ldots,\boldsymbol{a}_N;x_1,\ldots,x_N}_{*;\boldsymbol{c}_1,\ldots,\boldsymbol{c}_N}\right]=\prod_{s=1}^{N}F^{p_s}_{q_s}\left[{}^{\boldsymbol{a}_s;x_s}_{\boldsymbol{c}_s}\right]. \tag{5}$$

In the other extreme case, when i-parameters $\boldsymbol{a}_s$, $\boldsymbol{c}_s$ $(p_s=q_s=0;\ s=1,\ldots,N)$ are absent, the series (4) assumes a form of GHS-1 of the sum of the arguments x_s. Indeed, denoting the set of conditions

$$i_1\geqslant 0,\ldots,i_N\geqslant 0, \qquad i_1+i_2+\ldots+i_N=i, \tag{6}$$

as $[i_1, \ldots, i_N|i]$, we have

$$
\begin{aligned}
{}^NF^{p_0,0\ldots0}_{q_0,0\ldots0}\left[{}^{\boldsymbol{a}_0;*\ldots*;x_1,\ldots,x_N}_{\boldsymbol{c}_0;*\ldots*}\right] &= \sum_{i_1,\ldots,i_N} \frac{(\boldsymbol{a}_0)_{i_1+\ldots+i_N}}{(\boldsymbol{c}_0)_{i_1+\ldots+i_N}} \frac{x_1^{i_1}}{i_1!} \cdots \frac{x_N^{i_N}}{i_N!} \\
&= \sum_{i=0}^{\infty} \frac{(\boldsymbol{a}_0)_i}{(\boldsymbol{c}_0)_i} \frac{1}{i!} \sum_{[i_1\ldots i_N|i]} \frac{i!}{i_1!\,i_2!\ldots i_N!} x_1^{i_1} \ldots x_N^{i_N} \\
&= \sum_{i=0}^{\infty} \frac{(\boldsymbol{a}_0)_i}{(\boldsymbol{c}_0)_i} \frac{1}{i!} (x_1 + \ldots + x_N)^i \\
&= F^{p_0}_{q_0}\left[{}^{\boldsymbol{a}_0;x_1+\ldots+x_N}_{\boldsymbol{c}_0}\right].
\end{aligned} \tag{7}
$$

Amongst the standard hypergeometric series the following are the closest to the series (4): the Kampé de Fériet functions (Appell and Kampé de Fériet 1926, § XLIII) and the Lauricella functions (Appell and Kampé de Fériet 1926, §§ XXXVII–XXXIX). The case of the Kampé de Fériet functions corresponds in (4) to the following constraints:

$$N = 2, \qquad p_1 = p_2 = p, \qquad q_1 = q_2 = q.$$

The case of Lauricella functions arises when

$$p_s = p, \qquad q_s = q \qquad (s = 1, 2, \ldots, N), \tag{8}$$

$$p_0 + p = 2, \qquad q_0 + q = 1. \tag{9}$$

We shall not go into classification of the series ${}^NF$ which can be easily built up by analogy with the traditional classification (Appell and Kampé de Fériet 1926, § XLVIII)†. However, we shall specify the function

$${}^NF^{p_0,p}_{q_0,q} \equiv {}^NF^{p_0,p\ldots p}_{q_0,q\ldots q}, \tag{10}$$

which is most typical for applications, by the gluon (g) type (p_0/q_0), the individual (i) type (p/q) and the (total) type (p_0+p/q_0+q). Obviously, the particular type of the function ${}^NF$ (10) may be considered as the direct generalisation of either the Kampé de Fériet functions, for the case of arbitrary number of variables, or the Lauricella functions, for the case of arbitrary type.

Note that in our notations the Lauricella functions have the form

$$F_A = {}^NF^{1,1}_{0,1} \to F_2, \qquad F_B = {}^NF^{0,2}_{1,0} \to F_3, \qquad F_C = {}^NF^{2,0}_{0,1} \to F_4, \qquad F_D = {}^NF^{1,1}_{1,0} \to F_1,$$

where the correspondence with the Appell functions, into which the corresponding Lauricella functions transfer at $N = 2$, is indicated. All the Lauricella functions have the same type (2/1).

As has been mentioned in the introduction, the functions ${}^NF$ are encountered frequently, though implicitly, in physical applications (some examples are given below). Besides, these functions give an expedient framework which embraces numerous 'standard functions' and embody, at the same time, one of those classes of non-standard

† Note that classification of the double hypergeometric series by Horn's order has been criticised by Carlson (1976) who showed that the same function can be represented by two series of different orders. Therefore, 'the order of such a series is not a fundamental property' which implies that 'the theory of double hypergeometric series needs a new starting point' (Carlson 1976). The author is indebted to the referee for this indication.

hypergeometric functions which are 'nameless and have few recorded properties' (Wright *et al* 1977).

Note that the term 'generalised', applied to the function ${}^{N}F$, does not mean 'the most general'. The functions ${}^{N}F$, along with the Kampé de Fériet functions, correspond to the particular type of the 'general', i.e. satisfying the Horn criterion (Erdélyi 1953, § 5.7), hypergeometric series for which polynomials, entering into the ratio of contiguous coefficients of the series, are expanded into the product of linear functions of indexes. Such an analogy, apparently, would allow us to extend many results by Kampé de Fériet to the case of more general functions ${}^{N}F$. However, the approach used by Appell and Kampé de Fériet (1926, §§ XLVI–L), with the exception of some integral representations and the simplest reduction rule, is connected mainly with special problems of the theory of differential equations and is of restricted value for applications (recurrence relations, multiplication and addition theorems, connections with more simple functions, etc).

We establish here a number of simple properties of ${}^{N}F$ functions which, though following from elementary considerations, have not been, to the author's knowledge, formulated explicitly and, though not very profound, turn out to be of use for many purposes. First, we consider one practically important example that leads to ${}^{N}F$ functions of general type.

3. Reduction of the Lauricella functions with unit argument to ${}^{N}F$ functions. The differential identity for ${}^{N}F$

The functions ${}^{N}F$ of general type may arise, in particular, as a result of reduction of the Lauricella functions with unit argument. Such a situation takes place, for example, when the addition theorems for the Laguerre polynomials are used in molecular integrals with hydrogenic-like functions.

At the beginning, we shall use the following simple transformation of the ${}^{N}F$ function. By division of the set of summation indices in (4) into two sets $(i_1, \ldots, i_k)$ and $(i_{k+1}, \ldots, i_N)$, using the identity

$$(\boldsymbol{a}_0)_{i_1+\ldots+i_N} = (\boldsymbol{a}_0)_{i_1+\ldots+i_k}(\boldsymbol{a}_0+i_1+\ldots+i_k)_{i_{k+1}+\ldots+i_N},$$

where

$$\boldsymbol{a}_0+n = (a_0^1+n, \ldots, a_0^{p_0}+n), \tag{11}$$

and taking into account that summation over $i_{k+1}, \ldots, i_N$ gives rise to the function ${}^{N-k}F$, we obtain the identity of the Darling (1935) type:

$${}^{N}F = \sum_{i_1,\ldots,i_k} {}^{N-k}F^{p_0,p_{k+1}\ldots p_N}_{q_0,q_{k+1}\ldots q_N}\left[\begin{matrix}\boldsymbol{a}_0+i_1+\ldots+i_k;\boldsymbol{a}_{k+1},\ldots,\boldsymbol{a}_N;x_{k+1},\ldots,x_N\\ \boldsymbol{c}_0+i_1+\ldots+i_k;\boldsymbol{c}_{k+1},\ldots,\boldsymbol{c}_N\end{matrix}\right]\frac{(\boldsymbol{a}_0)_{i_1+\ldots+i_k}}{(\boldsymbol{c}_0)_{i_1+\ldots+i_k}}\prod_{s=1}^{k}\frac{(\boldsymbol{a}_s)_{i_s}}{(\boldsymbol{c}_s)_{i_s}}\frac{x_s^{i_s}}{i_s!}. \tag{12}$$

Although, for the reduction of the Lauricella functions, we need only a particular case of formula (12), we note here one general corollary of (12). Introducing operators $\delta_s = x_s\partial/\partial x_s$ and noting that for any analytical function

$$f(i_s)x_s^{i_s} = f(\delta_s)x_s^{i_s},$$

we obtain from (12) the operator identity

$${}^{N}F = {}^{N-k}F^{p_0,p_{k+1}\ldots p_N}_{q_0,q_{k+1}\ldots q_N}\left[\begin{matrix}\boldsymbol{a}_0+\delta_1+\ldots+\delta_k;\boldsymbol{a}_{k+1},\ldots,\boldsymbol{a}_N;x_{k+1},\ldots,x_N\\ \boldsymbol{c}_0+\delta_1+\ldots+\delta_k;\boldsymbol{c}_{k+1},\ldots,\boldsymbol{c}_N\end{matrix}\right]{}^{k}F. \tag{13}$$

Since (13) is valid for any k $(0 \leqslant k \leqslant N)$, letting $k = N-1$, we get the relation for the operator that 'generates' the function ${}^{N}F$ from ${}^{N-1}F$:

$$ {}^{N}F = F^{p_0+p_N}_{q_0+q_N}\left[\begin{smallmatrix} a_0+\delta_1+\ldots+\delta_{N-1}, a_N; x_N \\ c_0+\delta_1+\ldots+\delta_{N-1}, c_N \end{smallmatrix}\right]{}^{N-1}F. \tag{14} $$

Iterating (14), we obtain the identity

$$ {}^{N}F = F^{p_0+p_N}_{q_0+q_N}\left[\begin{smallmatrix} a_0+\delta_1+\ldots+\delta_{N-1}, a_N; x_N \\ c_0+\delta_1+\ldots+\delta_{N-1}, c_N \end{smallmatrix}\right] \ldots F^{p_0+p_2}_{q_0+q_2}\left[\begin{smallmatrix} a_0+\delta_1, a_2; x_2 \\ c_0+\delta_1, c_2 \end{smallmatrix}\right] F^{p_0+p_1}_{q_0+q_1}\left[\begin{smallmatrix} a_0, a_1; x_1 \\ c_0, c_1 \end{smallmatrix}\right] $$

that reveals a close connection of the function ${}^{N}F$ with N GHS-1 of the types (p_0+p_s/q_0+q_s), $s = 1, 2, \ldots, N$, respectively. In § 4 we shall obtain, however, an operator representation that forms a more convenient ground for studying ${}^{N}F$ functions, because it leads to more complete factorisation of the ${}^{N}F$ function and contains only one differential operator of simpler form.

Returning to (12), we assume that the function ${}^{N}F$ on the left-hand side is the Lauricella function with unit argument (for example, $x_N = 1$). Letting $k = N-1$ and taking into account that, in virtue of (8) and (9), the function ${}^{N-k}F$ in (12) takes the form

$$ F^{2}_{1}\left[\begin{smallmatrix} a_0+i_1+\ldots+i_{N-1}, a_N; 1 \\ c_0+i_1+\ldots+i_{N-1}, c_N \end{smallmatrix}\right], \tag{15} $$

equation (15) may be transformed by means of the Gauss theorem (Erdélyi 1953, equation 2.1 (14)). Depending on the type of the Lauricella function, the gluon index $I \equiv i_1 + \ldots + i_{n-1}$ appears in the right-hand part of the Gauss theorem in one of the following three forms: $\Gamma(\alpha+I)$, $\Gamma(\alpha-I)$ or $\Gamma(\alpha-2I)$. Using obvious identities

$$ \Gamma(\alpha-n) = \Gamma(\alpha)(\alpha-n)_n, \qquad (\alpha)_n = (-1)^n(-\alpha-n+1)_n, $$

as well as the formula of duplicating the Γ-function argument, we shall transform the contributions, depending on the index I, to the 'hypergeometric' form

$$ \Gamma(\alpha+I) = \Gamma(\alpha)(\alpha)_I, \qquad \Gamma(\alpha-I) = \Gamma(\alpha)(-1)^I(-\alpha+1)_I^{-1}, $$
$$ \Gamma(\alpha-2I) = \Gamma(\alpha)4^{-I}(-\alpha/2+1)_I^{-1}(-\alpha/2+\tfrac{1}{2})_I^{-1}. $$

Since in the case of F_B and F_D functions the partial contraction of gluon parameters of both numerator and denominator takes place, we obtain the following reduction formulae for F_A, F_B, F_C and F_D functions, respectively:

$$ {}^{N}F^{1,1}_{0,1}[x_N = 1] = F^{2}_{1}\left[\begin{smallmatrix} a_0^1, a_N^1; 1 \\ c_N^1 \end{smallmatrix}\right]{}^{N-1}F^{2,1}_{1,1}\left[\begin{smallmatrix} a_0^1, a_0^1-c_N^1+1; a_1^1, \ldots, a_{N-1}^1; x_1, \ldots, x_{N-1} \\ a_N^1+a_0^1-c_N^1+1; c_1^1, \ldots, c_{N-1}^1 \end{smallmatrix}\right], \tag{16} $$

$$ {}^{N}F^{0,2}_{1,0}[x_N = 1] = F^{2}_{1}\left[\begin{smallmatrix} a_N^1, a_N^2; 1 \\ c_0^1 \end{smallmatrix}\right]{}^{N-1}F^{1,2}_{2,0}\left[\begin{smallmatrix} c_0^1-a_N^1-a_N^2; a_1^1, a_1^2, \ldots, a_{N-1}^1, a_{N-1}^2; x_1, \ldots, x_{N-1} \\ c_0^1-a_N^1, c_0^1-a_N^2; *\ldots* \end{smallmatrix}\right], \tag{17} $$

$$ {}^{N}F^{2,0}_{0,1}[x_N = 1] = F^{2}_{1}\left[\begin{smallmatrix} a_0^1, a_0^2; 1 \\ c_N^1 \end{smallmatrix}\right]{}^{N-1}F^{4,0}_{2,1}\left[\begin{smallmatrix} a_0^1, a_0^2, a_0^1-c_N^1+1, a_0^2-c_N^1+1; *\ldots*; x_1/4, \ldots, x_{N-1}/4 \\ (a_0^1+a_0^2-c_N^1)/2+1, (a_0^1+a_0^2-c_N^1+1)/2; c_1^1, \ldots, c_{N-1}^1 \end{smallmatrix}\right], \tag{18} $$

$$ {}^{N}F^{1,1}_{1,0}[x_N = 1] = F^{2}_{1}\left[\begin{smallmatrix} a_0^1, a_N^1; 1 \\ c_0^1 \end{smallmatrix}\right]{}^{N-1}F^{1,1}_{1,0}\left[\begin{smallmatrix} a_0^1; a_1^1, \ldots, a_{N-1}^1; x_1, \ldots, x_{N-1} \\ c_0^1-a_N^1; *\ldots* \end{smallmatrix}\right]. \tag{19} $$

Thus, the reduction of ${}^{N}F$ functions of (2/1) type gives rise to ${}^{N-1}F$ functions of (3/2), (4/3) or (2/1) types, respectively. In the case of the function F_D the type of the function does not change, i.e. in case of other unit arguments (19) may be used again. When all the arguments are equal to 1, this leads to the formula

$$ {}^{N}F^{1,1}_{1,0}[1, \ldots, 1] = \prod_{s=1}^{N} F^{2}_{1}\left[\begin{smallmatrix} a_0^1, a_{N-s+1}^1; 1 \\ c_0^1-a_N^1-\ldots-a_{N-s+2}^1 \end{smallmatrix}\right] = F^{2}_{1}\left[\begin{smallmatrix} a_0^1, a_1^1+\ldots+a_N^1; 1 \\ c_0^1 \end{smallmatrix}\right], $$

which, with due respect to the Gauss theorem, is equivalent to the reduction formula of Appell and Kampé de Fériet (1926, equation (51) in § XXXVII). Other examples, that lead to the functions ${}^{N}F$ of the general type, are given in §§ 4, 5, 8.

4. Operator representation of the ${}^N F$ function

Supplying the arguments x_s with a common scaling factor, u, and introducing a summation index, i, according to (6), by analogy with the transformation (7), we have the following representation for the function ${}^N F$:

$$ {}^N_u F \equiv {}^N F^{p_0,p_1\ldots p_N}_{q_0,q_1\ldots q_N}\left[{}^{\boldsymbol{a}_0;\boldsymbol{a}_1,\ldots,\boldsymbol{a}_N;ux_1,\ldots,ux_N}_{\boldsymbol{c}_0;\boldsymbol{c}_1,\ldots,\boldsymbol{c}_N}\right] = \sum_{i=0}^{\infty} \frac{(\boldsymbol{a}_0)_i}{(\boldsymbol{c}_0)_i} u^i \, {}^N_i R, \tag{20} $$

$$ {}^N_i R \equiv {}^N_i R^{p_1\ldots p_N}_{q_1\ldots q_N}\left[{}^{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_N;x_1,\ldots,x_N}_{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_N}\right] = \sum_{[i_1\ldots i_N|i]} \prod_{s=1}^{N} \frac{(\boldsymbol{a}_s)_{i_s}}{(\boldsymbol{c}_s)_{i_s}} \frac{x_s^{i_s}}{i_s!}. \tag{21} $$

Writing down the product of N corresponding GHS-1 with a common scaling factor, t, in a form of GHS-N according to (5) and using then the transformation (20) for the resulting GHS-N, we obtain

$$ {}^N_t \Pi \equiv {}^N \Pi^{p_1\ldots p_N}_{q_1\ldots q_N}\left[{}^{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_N;tx_1,\ldots,tx_N}_{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_N}\right] = \prod_{s=1}^{N} F^{p_s}_{q_s}\left[{}^{\boldsymbol{a}_s;tx_s}_{\boldsymbol{c}_s}\right] = \sum_{i=0}^{\infty} t^i \, {}^N_i R, \tag{22} $$

and hence

$$ {}^N_i R = \frac{1}{i!} \frac{\partial^i}{\partial t^i} {}^N_t \Pi|_{t=0}. \tag{23} $$

Note that it follows from (20) and (22) that the integration of functions ${}^N_u F$, ${}^N_u \Pi$ by the variable u with such a weight function, $\varphi(u)$, that the integral

$$ \int_a^b \mathrm{d}u \, \varphi(u) u^i = \frac{(\boldsymbol{a}'_0)_i}{(\boldsymbol{c}'_0)_i} k^i q $$

has a Γ-product form (k and q are constants independent of i), leads to the function ${}^N_k F$ of higher type. Such a situation takes place, in particular, for the coefficients of expansions of ${}^N_u F$ or ${}^N_u \Pi$ functions over the Laguerre polynomials and 'shifted' Jacobi polynomials, as well as over the generalised analogues of these polynomials (Luke 1975, ch II).

Substitution of (23) into (20) yields, with due respect for definition (1), the following operator representation of the function ${}^N F$:

$$ {}^N_u F = F^{p_0}_{q_0}\left[{}^{\boldsymbol{a}_0;u\partial/\partial t}_{\boldsymbol{c}_0}\right] {}^N_t \Pi|_{t=0} $$

or

$$ {}^N F^{p_0,p_1\ldots p_N}_{q_0,q_1\ldots q_N}\left[{}^{\boldsymbol{a}_0;\boldsymbol{a}_1,\ldots,\boldsymbol{a}_N;ux_1,\ldots,ux_N}_{\boldsymbol{c}_0;\boldsymbol{c}_1,\ldots,\boldsymbol{c}_N}\right] = F^{p_0}_{q_0}\left[{}^{\boldsymbol{a}_0;u\partial/\partial t}_{\boldsymbol{c}_0}\right] F^{p_1}_{q_1}\left[{}^{\boldsymbol{a}_1;tx_1}_{\boldsymbol{c}_1}\right] \ldots F^{p_N}_{q_N}\left[{}^{\boldsymbol{a}_N;tx_N}_{\boldsymbol{c}_N}\right]|_{t=0}. \tag{24} $$

The remarkable feature of the representation (24) is a complete multiplicative factorisation of the function ${}^N_u F$ in variables $u, x_1, \ldots, x_N$ which leads to numerous analytical corollaries. In the case of GHS-1 equation (24) assumes the form ($p = p_0 + p_1$, $q = q_0 + q_1$)

$$ F^p_q\left[{}^{\boldsymbol{a};ux}_{\boldsymbol{c}}\right] \equiv F^{p_0+p_1}_{q_0+q_1}\left[{}^{\boldsymbol{a}_0,\boldsymbol{a}_1;ux}_{\boldsymbol{c}_0,\boldsymbol{c}_1}\right] = F^{p_0}_{q_0}\left[{}^{\boldsymbol{a}_0;u\partial/\partial t}_{\boldsymbol{c}_0}\right] F^{p_1}_{q_1}\left[{}^{\boldsymbol{a}_1,tx}_{\boldsymbol{c}_1}\right]|_{t=0}, \tag{25} $$

where

$$ \boldsymbol{a}_0 = a^{1\ldots p_0}, \qquad \boldsymbol{a}_1 = a^{p_0+1\ldots p}, \qquad \boldsymbol{c}_0 = c^{1\ldots q_0}, \qquad \boldsymbol{c}_1 = c^{q_0+1\ldots q}. $$

Thus (24) not only makes it possible to reduce ${}^N F$ series to ${}^1 F$ series, but allows also the further reduction of GHS-1 of a complicated type to GHS-1 of simpler type.

Using the differentiation formula for GHS-1 (Luke 1975, equation 5.2.2 (1))

$$\frac{\partial^n}{\partial x^n} F^p_q\left[{a;tx \atop c}\right] = t^n \frac{(\boldsymbol{a})_n}{(\boldsymbol{c})_n} F^p_q\left[{a+n;tx \atop c+n}\right], \tag{26}$$

where the symbol $\boldsymbol{a}+n$ is explained in (11), we obtain by means of (24)

$$\frac{\partial^{n_0}}{\partial x_1^{n_1} \ldots \partial x_N^{n_N}} {}^N_uF = \prod_{s=1}^{N} \frac{(\boldsymbol{a}_s)_{n_s}}{(\boldsymbol{c}_s)_{n_s}} F_0(ud(t)) t^{n_0} F_1^{[n_1]}(x_1 t) \ldots F_N^{[n_N]}(x_N t), \tag{27}$$

where $n_0 \equiv n_1 + \ldots + n_N$, $d(t) \equiv \partial/\partial t$, and the symbol $[n_s]$ denotes the shift of all the parameters of both numerator and denominator of the corresponding GHS-1 by the value n_s. Writing down an arbitrary function $F(d)$, where $d = d(t)$, as a series in powers of the operator d, and using for $i < n$ the commutation relation

$$d^i t^n = i!\, t^{n-i} L_i^{n-i}(-t_{②} d_{①}),$$

and for $i \geqslant n$ the commutation relation

$$d^i t^n = n!\, L_n^{i-n}(-t_{②} d_{①}) d^{i-n},$$

where $(t_{②} d_{①})^q \equiv t^q d^q$, and also taking into account the expression of the Laguerre polynomials in terms of the Kummer function, Φ, we obtain

$$F(d) t^n f(t)|_{t=0} = F^{(n)}(d) f(t)|_{t=0}, \tag{28}$$

where $F^{(n)}(z) \equiv (\partial^n/\partial z^n) F(z)$. Applying (28) to expression (27) with due respect for the formula (26), we obtain the differentiation rule for the function ^NF:

$$\frac{\partial^{n_0}}{\partial x_1^{n_1} \ldots \partial x_N^{n_N}} {}^N_uF = u^{n_0} \prod_{\sigma=0}^{N} \frac{(\boldsymbol{a}_\sigma)_{n_\sigma}}{(\boldsymbol{c}_\sigma)_{n_\sigma}} {}^NF^{p_0, p_1 \ldots p_N}_{q_0, q_1 \ldots q_N}\left[{\boldsymbol{a}_0 + n_0; \boldsymbol{a}_1 + n_1, \ldots, \boldsymbol{a}_N + n_N; ux_1, \ldots, ux_N \atop \boldsymbol{c}_0 + n_0; \boldsymbol{c}_1 + n_1, \ldots, \boldsymbol{c}_N + n_N}\right].$$

In the following we give some less trivial applications of the operator representation (24).

5. The generating function for the series ^NF. Generalisation of the Chaundy expansions

Equation (24) shows the close link of the function ^NF with the product of N hypergeometric series of one variable. We shall give here one more aspect of such a link that leads to a far reaching generalisation of four Chaundy (1943) expansions of particular type (see also Erdélyi 1953, equations 4.3 (12–15)). At the same time, this gives rise to a simple addition theorem for the product of N GHS-1 and yields one more important example of practical interest that leads to the functions ^NF.

Consider the quantities ^N_iR (21) which are the coefficients in the expansions of the functions ^N_uF (20) and $^N_t\Pi$ (22) in powers of parameters u and t, respectively. Since the summation variables i_s $(s = 1, \ldots, N)$, in (21) obey the linear equation (6), one of them may be excluded from the summation if we express it in terms of the other $N-1$ variables, i_s, and the variable i. For example,

$$i_N = i - I, \qquad I = i_1 + i_2 + \ldots + i_{N-1}.$$

The contributions depending on the index I appear in (21) in the form

$$[(\boldsymbol{a}_N)_{i-I}/(\boldsymbol{c}_N)_{i-I}][(i-I)!]^{-1} x_N^{i-I}. \tag{29}$$

As in § 3, we transform this expression in such a way that the contributions depending on I would assume a 'hypergeometric form' of the Pochhammer symbols. Using the transformation

$$(a)_{i-I}=(a)_i(-1)^I(1-a-i)_I^{-1} \tag{30}$$

and its particular version

$$1/(i-I)!=(1)_{i-I}^{-1}=(1/i!)(-1)^I(-1)_I, \tag{31}$$

we obtain the necessary expression:

$$\frac{(\boldsymbol{a}_N)_{i-I}}{(\boldsymbol{c}_N)_{i-I}}\frac{x_N^{i-I}}{(i-I)!}=\frac{(\boldsymbol{a}_N)_i}{(\boldsymbol{c}_N)_i}\frac{x_N^i}{i!}\frac{(-i)_I(1-i-\boldsymbol{c}_N)_I}{(1-i-\boldsymbol{a}_N)_I}\left[(-1)^{q_N+1-p_N}\frac{1}{x_N}\right]^I.$$

As a result, the quantity ${}^N_iR$ assumes the form of a generalised hypergeometric series of $N-1$ variables:

$${}^N_iR=\frac{(\boldsymbol{a}_N)_i}{(\boldsymbol{c}_N)_i}\frac{x_N^i}{i!}\,{}^{N-1}F^{q_N+1,p_1\dots p_{N-1}}_{p_N,q_1\dots q_{N-1}}\left[\begin{smallmatrix}-i,1-i-\boldsymbol{c}_N;\boldsymbol{a}_1,\dots,\boldsymbol{a}_{N-1};\sigma_Nx_1/x_N,\dots,\sigma_Nx_{N-1}/x_N\\ 1-i-\boldsymbol{a}_N;\boldsymbol{c}_1,\dots,\boldsymbol{c}_{N-1}\end{smallmatrix}\right], \tag{32}$$

where $\sigma_N=(-1)^{Q_N}$ and $Q_N=q_N+1-p_N$.

This means, in virtue of (22), that the product of N generalised hypergeometric series serves the generating function for the function ${}^{N-1}F$ which is presented by (32):

$$\begin{aligned}{}^N_t\Pi&=F^{p_1}_{q_1}\left[\begin{smallmatrix}\boldsymbol{a}_1;x_1t\\ \boldsymbol{c}_1\end{smallmatrix}\right]\dots F^{p_{N-1}}_{q_{N-1}}\left[\begin{smallmatrix}\boldsymbol{a}_{N-1};x_{N-1}t\\ \boldsymbol{c}_{N-1}\end{smallmatrix}\right]F^{p_N}_{q_N}\left[\begin{smallmatrix}\boldsymbol{a}_N;x_Nt\\ \boldsymbol{c}_N\end{smallmatrix}\right]\\ &=\sum_{i=0}^{\infty}\frac{(\boldsymbol{a}_N)_i}{(\boldsymbol{c}_N)_i}\frac{x_N^i}{i!}\,{}^{N-1}F^{q_N+1,p_1\dots p_{N-1}}_{p_N,q_1\dots q_{N-1}}\left[\begin{smallmatrix}-i,1-i-\boldsymbol{c}_N;\boldsymbol{a}_1,\dots,\boldsymbol{a}_{N-1};\sigma_Nx_1/x_N,\dots,\sigma_Nx_{N-1}/x_N\\ 1-i-\boldsymbol{a}_N;\boldsymbol{c}_1,\dots,\boldsymbol{c}_{N-1}\end{smallmatrix}\right]t^i.\end{aligned} \tag{33}$$

This circumstance alone justifies the introduction of ${}^NF$ functions, to say nothing of the other examples. In the case $N=2$ the function ${}^{N-1}F$ on the right-hand side of (33) has a form of GHS of one variable. This allows us to write down the expansion of a product of two GHS's in a form

$$F^{p_1}_{q_1}\left[\begin{smallmatrix}\boldsymbol{a}_1;tx_1\\ \boldsymbol{c}_1\end{smallmatrix}\right]F^{p_2}_{q_2}\left[\begin{smallmatrix}\boldsymbol{a}_2;tx_2\\ \boldsymbol{c}_2\end{smallmatrix}\right]=\sum_{i=0}^{\infty}\frac{(\boldsymbol{a}_2)_i}{(\boldsymbol{c}_2)_i}\frac{x_2^i}{i!}F^{p_1+q_2+1}_{q_1+p_2}\left[\begin{smallmatrix}-i,1-i-\boldsymbol{c}_2,\boldsymbol{a}_1;\sigma_2x_1/x_2\\ 1-i-\boldsymbol{a}_2,\boldsymbol{c}_1\end{smallmatrix}\right]t^i.$$

As particular cases, from this expansion follow four Chaundy (1943) expansions for the products of two GHS's of a special type: $F^0_1F^0_1$, $F^1_1F^1_1$, $F^2_1F^2_1$, $F^2_0F^2_0$.

As other particular cases of formula (33), the generating functions for the Lauricella functions F_C and F_D are easily obtained:

$$\begin{aligned}F^0_1\left[\begin{smallmatrix}*;tx_1\\ c_1\end{smallmatrix}\right]\dots F^0_1\left[\begin{smallmatrix}*;tx_N\\ c_N\end{smallmatrix}\right]&=\sum_{i=0}^{\infty}\frac{x_N^i}{(c_N)_ii!}\,{}^{N-1}F^{2,0}_{0,1}\left[\begin{smallmatrix}-i,1-i-c_N;*\dots*;x_1/x_N,\dots,x_{N-1}/x_N\\ *;c_1,\dots,c_{N-1}\end{smallmatrix}\right]t^i,\\ F^1_0\left[\begin{smallmatrix}a_1;tx_1\\ *\end{smallmatrix}\right]\dots F^1_0\left[\begin{smallmatrix}a_N;tx_N\\ *\end{smallmatrix}\right]&=\sum_{i=0}^{\infty}(a_N)_i\frac{x_N^i}{i!}\,{}^{N-1}F^{1,1}_{1,0}\left[\begin{smallmatrix}-i;a_1,\dots,a_{N-1};x_1/x_N,\dots,x_{N-1}/x_N\\ 1-i-a_N;*\dots*\end{smallmatrix}\right]t^i.\end{aligned} \tag{34}$$

Note that in the case of non-positive integers $a_1,\dots,a_N$ equation (34) yields an explicit expression for the coefficients of a polynomial in terms of its roots, i.e. an alternative formulation of the Vietta theorem. Formula (34) turns out to be useful also for the derivation of one of the versions of an addition theorem for the Laguerre polynomials.

Note that the dependence of the function ${}^{N-1}F$ in (33) on its arguments is of polynomial type, since one of the gluon numerators is a non-positive integer. The function ${}^{N}F$ turns out tó be polynomial also in the case when each of the sets of i-numerators a_s, $s = 1, 2, \ldots, N$, contains a non-negative integer $(-n_s)$. In this case, by analogy with the transformation of a terminating series, F^p_q (Luke 1975, equation 5.2.1 (5)), the polynomial ${}^{N}F$ can by ordered by descending rather than ascending powers of arguments. To this end, we replace variables i_s on the right-hand side of the definition (4) by the new variables $j_s = n_s - i_s$, and make use of tranformations (30), (31). As a result, we obtain

$$
\begin{aligned}
{}^{N}F^{p_0,p_1+1\ldots p_N+1}_{q_0,q_1\ldots q_N}\left[{}^{\boldsymbol{a}_0;-n_1,\boldsymbol{a}_1,\ldots,-n_N,\boldsymbol{a}_N;x_1,\ldots,x_N}_{\boldsymbol{c}_0;\boldsymbol{c}_1,\ldots,\boldsymbol{c}_N}\right] &= \frac{(\boldsymbol{a}_0)_{n_1+\ldots+n_N}}{(\boldsymbol{c}_0)_{n_1+\ldots+n_N}} \prod_{s=1}^{N} (-x_s)^{n_s} \frac{(\boldsymbol{a}_s)_{n_s}}{(\boldsymbol{c}_s)_{n_s}} \\
&\times {}^{N}F^{q_0,q_1+1\ldots q_N+1}_{p_0,p_1\ldots p_N}\left[{}^{-\boldsymbol{c}_0-n_1-\ldots-n_N+1;-n_1,-\boldsymbol{c}_1-n_1+1,\ldots,-n_N,-\boldsymbol{c}_N-n_N+1;\delta_1/x_1,\ldots,\delta_N/x_n}_{-\boldsymbol{a}_0-n_1-\ldots-n_N+1;-\boldsymbol{a}_1-n_1+1,\ldots,-\boldsymbol{a}_N-n_N+1}\right],
\end{aligned}
\tag{35}
$$

where $\delta_s = (-1)^{p_0+q_0+p_s+q_s}$.

6. Recurrence relations

The operator representation (24) allows us to obtain recurrence relations for the series ${}^{N}F$ with the help of recurrence relations for the series ${}^{1}F$ appearing on the right-hand side of (24). In their turn, the recurrence relations for the functions ${}^{1}F$ can be derived by means of the operator representation (25). In some special cases this gives a useful alternative to the Rainville (1945, 1960) approach†.

Two simple forms of factorisation of the series ${}^{1}F$, based on the representation (25), have the form

$$
F^p_q\left[{}^{a^{1\ldots p};x}_{c^{1\ldots q}}\right] = F^1_0\left[{}^{a^{\nu};\partial/\partial t}_{*}\right] F^{p-1}_q\left[{}^{a^{1}\ldots a^{\nu-1}a^{\nu+1}\ldots a^{p};tx}_{c^{1\ldots q}}\right]_{t=0}, \tag{36}
$$

$$
F^p_q\left[{}^{a^{1\ldots p};x}_{c^{1\ldots q}}\right] = F^0_1\left[{}^{*;\partial/\partial t}_{c^{\mu}}\right] F^p_{q-1}\left[{}^{a^{1\ldots p};tx}_{c^{1}\ldots c^{\mu-1}c^{\mu+1}\ldots c^{q}}\right]_{t=0}. \tag{37}
$$

Formulae (36) and (37) allow us to use for studying the series F^p_q of general type the properties of the 'elementary' series F^1_0 and F^0_1 connected, respectively, with the binomial function and the Bessel function (Erdélyi 1953, equation 7.2 (3)). Using this connection, we have the following recurrence relations for the series F^1_0 and F^0_1:

$$
F^1_0\left[{}^{a;z}_{*}\right] - F^1_0\left[{}^{a+1;z}_{*}\right] + F^1_0\left[{}^{a+1;z}_{*}\right] z = 0, \tag{38}
$$

$$
c(c-1)F^0_1\left[{}^{*;z}_{c-1}\right] - c(c-1)F^0_1\left[{}^{*;z}_{c}\right] - F^0_1\left[{}^{*;z}_{c+1}\right] z = 0. \tag{39}
$$

Using the relations (38), (39) in (36), (37) and applying again the formula (25), we obtain

$$
a^{\nu} F^p_q\left[{}^{\boldsymbol{a};x}_{\boldsymbol{c}}\right] - a^{\nu} F^p_q\left[{}^{\ldots a^{\nu}+1\ldots;x}_{\boldsymbol{c}}\right] + x\frac{a^1 \ldots a^p}{c^1 \ldots c^q} F^p_q\left[{}^{\boldsymbol{a}+1;x}_{\boldsymbol{c}+1}\right] = 0, \tag{40}
$$

$$
(c^{\mu}-1) F^p_q\left[{}^{\boldsymbol{a};x}_{\ldots c^{\mu}-1\ldots}\right] - (c^{\mu}-1) F^p_q\left[{}^{\boldsymbol{a};x}_{\boldsymbol{c}}\right] - x\frac{a^1 \ldots a^p}{c^1 \ldots c^q} F^p_q\left[{}^{\boldsymbol{a}+1;x}_{\boldsymbol{c}+1}\right] = 0. \tag{41}
$$

† Rainville derived a complete set of recurrence relations for a *special* class of *contiguous* ${}_pF_q$'s. In this connection an interesting question arises whether *all* the recurrence relations or at least all those for contiguous ${}_pF_q$'s can be obtained from the operator representation (25). The author is indebted to the referee who indicated this problem.

Eliminating the last terms from (40) and (41), we arrive at the recurrence relation

$$(a^{\nu}-c^{\mu}+1)F_q^p\left[\begin{smallmatrix}\boldsymbol{a};x\\ \boldsymbol{c}\end{smallmatrix}\right]-a^{\nu}F_q^p\left[\begin{smallmatrix}\ldots a^{\nu}+1\ldots;x\\ \boldsymbol{c}\end{smallmatrix}\right]+(c^{\mu}-1)F_q^p\left[\begin{smallmatrix}\boldsymbol{a};x\\ \ldots c^{\mu}-1\ldots\end{smallmatrix}\right]=0. \tag{42}$$

In contrast to (40) and (44), only two parameters, a^{ν} and c^{μ}, change their values in (42). Writing down (40) for some value ν' different from ν and eliminating the last term in (40), we obtain the relation

$$(a^{\nu}-a^{\nu'})F_q^p\left[\begin{smallmatrix}\boldsymbol{a};x\\ \boldsymbol{c}\end{smallmatrix}\right]-a^{\nu}F_q^p\left[\begin{smallmatrix}\ldots a^{\nu}+1\ldots;x\\ \boldsymbol{c}\end{smallmatrix}\right]+a^{\nu'}F_q^p\left[\begin{smallmatrix}\ldots a^{\nu'}+1\ldots;x\\ \boldsymbol{c}\end{smallmatrix}\right]=0, \tag{43}$$

which corresponds to change of two parameters, a^{ν} and $a^{\nu'}$. A similar procedure in the case of (41) results in the relation

$$(c^{\mu}-c^{\mu'})F_q^p\left[\begin{smallmatrix}\boldsymbol{a};x\\ \boldsymbol{c}\end{smallmatrix}\right]+(c^{\mu}-1)F_q^p\left[\begin{smallmatrix}\boldsymbol{a};x\\ \ldots c^{\mu}-1\ldots\end{smallmatrix}\right]-(c^{\mu'}-1)F_q^p\left[\begin{smallmatrix}\boldsymbol{a};x\\ \ldots c^{\mu'}-1\ldots\end{smallmatrix}\right]=0, \tag{44}$$

in which only two parameters, c^{μ} and $c^{\mu'}$ are changed. Thus, (42), (43) and (44) correspond to three cases of various types, when one of two changing parameters is numerator and the other one is denominator, or both of them are either numerators or denominators simultaneously.

Each of these equations can be used either for the gluon function, $F_{q_0}^{p_0}$, or for any of the series $F_{q_s}^{p_s}$ $(s=1,2,\ldots,N)$ in (24). For example, applying (40) for $F_{q_0}^{p_0}$ in (24) and using the differentiation rule (26) we have the following recurrence relation for the series ${}^N F$ (we write down only the changing parameters):

$$a_0^{\nu}\,{}^N_u F - a_0^{\nu}\,{}^N_u F\left[\begin{smallmatrix}\ldots a_0^{\nu}+1\ldots\\ \ldots\end{smallmatrix}\right]+u\frac{a_0^1\ldots a_0^{p_0}}{c_0^1\ldots c_0^{q_0}}\sum_{s=1}^{N}x_s\frac{a_s^1\ldots a_s^{p_s}}{c_s^1\ldots c_s^{q_s}}\,{}^N_u F\left[\begin{smallmatrix}\boldsymbol{a}_0+1;\ldots\boldsymbol{a}_s+1\ldots\\ \boldsymbol{c}_0+1;\ldots\boldsymbol{c}_s+1\ldots\end{smallmatrix}\right]=0,$$

where $\nu=1,\ldots,p_0$. Applying the same equation for some series $F_{q_s}^{p_s}$ in (24) and then taking into account relation (28), we obtain

$$a_s^{\nu}\,{}^N_u F - a_s^{\nu}\,{}^N_u F\left[\begin{smallmatrix}\ldots a_s^{\nu}+1\ldots\\ \ldots\end{smallmatrix}\right]+x_s u\frac{(\boldsymbol{a}_s)_1(\boldsymbol{a}_0)_1}{(\boldsymbol{c}_s)_1(\boldsymbol{c}_0)_1}\,{}^N_u F\left[\begin{smallmatrix}\boldsymbol{a}_0+1;\ldots\boldsymbol{a}_s+1\ldots\\ \boldsymbol{c}_0+1;\ldots\boldsymbol{c}_s+1\ldots\end{smallmatrix}\right]=0.$$

A similar procedure for (41) gives the following recurrence relations:

$$(c_0^{\mu}-1)\,{}^N_u F\left[\begin{smallmatrix}\ldots\\ \ldots c_0^{\mu}-1\ldots\end{smallmatrix}\right]-(c_0^{\mu}-1)\,{}^N_u F-u\frac{(\boldsymbol{a}_0)_1}{(\boldsymbol{c}_0)_1}\sum_{s=1}^{N}x_s\frac{(\boldsymbol{a}_s)_1}{(\boldsymbol{c}_s)_1}\,{}^N_u F\left[\begin{smallmatrix}\boldsymbol{a}_0+1;\ldots\boldsymbol{a}_s+1\ldots\\ \boldsymbol{c}_0+1;\ldots\boldsymbol{c}_s+1\ldots\end{smallmatrix}\right]=0,$$

$$(c_s^{\mu}-1)\,{}^N_u F\left[\begin{smallmatrix}\ldots\\ \ldots c_s^{\mu}-1\ldots\end{smallmatrix}\right]-(c_s^{\mu}-1)\,{}^N_u F-x_s u\frac{(\boldsymbol{a}_0)_1(\boldsymbol{a}_s)_1}{(\boldsymbol{c}_0)_1(\boldsymbol{c}_s)_1}\,{}^N_u F\left[\begin{smallmatrix}\boldsymbol{a}_0+1;\ldots\boldsymbol{a}_s+1\ldots\\ \boldsymbol{c}_0+1;\ldots\boldsymbol{c}_s+1\ldots\end{smallmatrix}\right]=0.$$

We shall not write down the corresponding analogues of (42), (43), (44), since they differ from the initial equations only in substitution of the values of a_0^{ν}, $a_0^{\nu'}$, c_0^{μ}, $c_0^{\mu'}$ or a_s^{ν}, $a_s^{\nu'}$, c_s^{μ}, $c_s^{\mu'}$ instead of a^{ν}, $a^{\nu'}$, c_{μ}, c'_{μ} and in replacement of the symbol F_q^p by ${}^N F$.

7. Expansion of function ${}^N F$ in terms of simpler functions. Generalisation of the Bailey expansion

The operator representation (24) provides also a more systematic basis for obtaining various expansions and addition theorems. Indeed, using some transformation of the product of GHS-1 in (24) and applying the operator $F(u\partial/\partial t)$ to the resulting expression, one may easily obtain various alternative representations for the function ${}^N F$.

As an example, we obtain the expansion of function ${}^{N}F$ of arbitrary g-type over functions ${}^{N}F$ of g-type (2/0) which coincide with the Lauricella functions, F_C (18), for the case where all i-types of the initial function ${}^{N}F$ are equal to (0/1).

First, we shall obtain an expansion of the product of N arbitrary GHS-1 over the functions F_1^0, which generalises the corresponding Bailey (1935b)† expansion for the product of two GHS-1 of (0/1)-types (see also equation 7.15(7) in Erdélyi 1953). Writing down the product of N GHS-1 in the form of equation (22), then multiplying both sides by the quantity $t^{\gamma/2}$, using the Neumann series for the function $\sqrt{t}^{\gamma+2i}$, making obvious substitutions and reordering of summation variables and then using the same transformations as in (30) and (31) with due account of the formula (20), we obtain the necessary generalisation of the Bailey theorem:

$$ {}_{u}^{N}\Pi = \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+m)}{m!\,\Gamma(\gamma+2m)} {}^{N}F_{0,q_1\ldots q_N}^{2,p_1\ldots p_N}\left[{}^{-m,\gamma+m;\boldsymbol{a}_1,\ldots,\boldsymbol{a}_N;x_1,\ldots,x_N}_{*;\boldsymbol{c}_1,\ldots,\boldsymbol{c}_N}\right](-t)^m F_1^0\left[{}^{*;t}_{\gamma+2m+1}\right]. \tag{45} $$

In the case $N=2$, $p_1=p_2=0$, $q_1=q_2=1$ the expansion (45) is equivalent to Bailey's (1935b) expansion (3.1).

By virtue of relations (28), (26)

$$ F_{q_0}^{p_0}\left[{}^{\boldsymbol{a}_0;u\partial/\partial t}_{\boldsymbol{c}_0}\right] t^m F_{q_1}^{p_1}\left[{}^{\boldsymbol{a}_1;tx}_{\boldsymbol{c}_1}\right]\Big|_{t=0} = u^m \frac{(\boldsymbol{a}_0)_m}{(\boldsymbol{c}_0)_m} F_{q_0+q_1}^{p_0+p_1}\left[{}^{\boldsymbol{a}_0+m,\boldsymbol{a}_1;ux}_{\boldsymbol{c}_0+m,\boldsymbol{c}_1}\right]. \tag{46} $$

Using (45) and (46) in (24), we obtain finally

$$ \begin{aligned} {}_{u}^{N}F = \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\gamma+m)(\boldsymbol{a}_0)_m}{m!\,\Gamma(\gamma+2m)(\boldsymbol{c}_0)_m} & {}^{N}F_{0,q_1\ldots q_N}^{2,p_1\ldots p_N}\left[{}^{-m,\gamma+m;\boldsymbol{a}_1,\ldots,\boldsymbol{a}_N;x_1,\ldots,x_N}_{*;\boldsymbol{c}_1,\ldots,\boldsymbol{c}_N}\right] \\ & \times u^m F_{q_0+1}^{p_0}\left[{}^{\boldsymbol{a}_0+m;u}_{\boldsymbol{c}_0+m,\gamma+2m+1}\right]. \end{aligned} \tag{47} $$

Equation (47) may be interpreted either as expansion in terms of the functions ${}^{N}F$ of g-type (2/0), or as expansion over the functions $F_{q_0+1}^{p_0}(u)$, i.e. as the 'argument's multiplication theorem' associated with the scaling transformation of the function ${}^{N}F$.

8. Concluding remarks

We did not intend either to carry out some complete investigation of the general properties of the functions ${}^{N}F$, or to consider the detailed applications of these functions to some specific problem of mathematical physics. Nevertheless, there are clear indications of numerous possible applications of these functions. Indeed, it is clear from the above examples that the functions ${}^{N}F$ comprise not only various types of standard hypergeometric series, but also those series which result from the typical operations (such as integration, reduction, addition theorems, etc) over the standard series. In particular, these functions are useful in evaluation of multicentre integrals in variational calculations of molecular electron structure. For example, the coefficients in addition theorems for the Laguerre polynomials have a compact expression in terms of these functions. This, in turn, considerably facilitates the calculation of multicentre integrals with hydrogenic-like functions.

† Bailey formulates such an expansion in terms of the Bessel functions rather than the series F_1^0.

Note one more important class of the problems leading to the functions ${}^{N}F$ of a general type. Let $\tilde{\varphi}_n$ be the adjacent function to a function φ_n, that is

$$\exp(ax) = \sum_n \tilde{\varphi}_n(a)\varphi_n(x).$$

Using the identity

$$f(a) = \exp(\mathrm{d}/\mathrm{d}t)f(t)|_{t=0}$$

that follows, for example, from the Taylor series for the function $f(a+t)$, one may easily show that the coefficients of the formal expansion over the adjacent system of functions

$$f(a) = \sum_n c_n\tilde{\varphi}(a/u)$$

have the form

$$c_n = \varphi_n(u\partial/\partial t)f(t)|_{t=0}.$$

This means that the coefficients in the expansion of the function $f(a) \equiv {}^{N}_{a}\Pi$ over a complete set $\tilde{\varphi}_n(a/u)$ coincide with the functions ${}^{N}_{u}F$ (24), if

$$\varphi_n(x) = F^{p_0}_{q_0}\left[{}^{\boldsymbol{a}_0;x}_{\boldsymbol{c}_0}\right],$$

where the parameters $\boldsymbol{a}_0$, $\boldsymbol{c}_0$ are supposed to be linearly dependent on the index n. A number of similar examples is formed by expansions of the exponential function into the Fields and Wimp (1963) series (see also Luke 1975, ch II).

As far as more detailed and complete study of the functions ${}^{N}_{u}F$ is concerned, the above examples show that the operator representation (24) may serve as a convenient basis for deriving any properties of these functions which may be needed in applications. Indeed, equation (24) ensures that any property of the function ${}^{N}_{u}F$ is a consequence of the properties of generalised hypergeometric series of one variable, many of which were studied in detail. In those cases when the necessary information is absent, the specific version of an operator representation (25) allows us to reduce a corresponding hypergeometric series or one variable to the series of a simpler type: eventually, to the series of binomial (F^1_0) and Bessel (F^0_1) type, as takes place, for example, in (36) and (37).

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